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REGULAR AND ANOMALOUS REGIMES OF GAS–LIQUID FLOW THROUGH A CHANNEL CONTRACTION

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1. Introduction. The problem of the adequate choice of boundary conditions is very real in the modeling of multiphase flows in a channel of finite length. The condition that flow be critical in the neighborhood of the minimum section of the channel is widely used in one-dimensional hyperbolic flow models It allows us to describe the upstream influence of a local contraction. This approach, originating in the hydraulics of open channel flows, is also well-known in gas dynamics and is used in modeling of transonic nozzle flows in a channel approximation. An obstacle has control over the upstream flow if it provides a transition from subcritical (subsonic) to supercritical (supersonic) flow. In a channel of variable cross-section, such a regular flow regime holds for one-dimensional flows of shallow water and normal gas. But in the case of more complicated models of a multiphase or multicomponent fluid, anomalous flow regimes can arise for which an obstacle "supports" the propagation of upstream disturbances of finite amplitude and, at the same time, the flow through the channel contraction is totally supercritical. For two-layer shallow water flow above an uneven bottom, such anomalous regimes were experimentally found in [1, 2] and investigated in [3].

The aim of this work is to investigate theoretically, in the framework of the channel approximation, the nonsteady wave motion of a barotropic compressible fluid with a nonconvex equation of state in a neighborhood of local contraction of the channel. For large times, the problem is reduced to a self-similar one and can be divided into two stages, each of which has been well-studied to date: the wave structure for the problem of discontinuity decay and the structure of steady flows in a channel of variable cross-section for gases with a nonconvex equation of a state. The first problem has been studied in [4–7]. Considerable study has recently been given to steady flows of dense gases through a Laval nozzle [8–10] in connection with possible applications in engineering problems.

The author does not pose the problem of complete classification of possible flow regimes generated by a sudden contraction of a channel. Most of the attention has been concentrated on the detection and explanation (based on the analysis of the nonstationary problem) of the causes of anomalous flow regimes in the neighborhood of an obstacle.

2. The Equation of State of a Gas–Liquid Mixture. One of the simplest models of a gas–liquid medium is a one-velocity one-pressure model of joint motion of the gas and incompressible liquid component. One-dimensional motions are governed by the following system of equations:

$$\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2 + p)_x = 0. \tag{2.1}$$

Here ρ is the density, u is the medium velocity, and the equation of state $p = g(\tau)$ $(\tau = 1/\rho)$ is found from equality of the pressures in the liquid and gas components. If the void concentration is small, then the liquid can be considered to be a thermostat and the process of void-concentration variation to be isothermal: $p = f(V, T)$ ($T \equiv$ const, V is the specific gas volume). Thus, for the inert gas component (the mass concentration $\lambda \equiv$ const), the effective equation of state $p = g(\tau) = f(V,T)$ is completely specified by the compressibility of the gas phase, since the specific liquid volume $\tau = \lambda V + (1 - \lambda)\tau_f$. Moreover, on changing to the Lagrange mass

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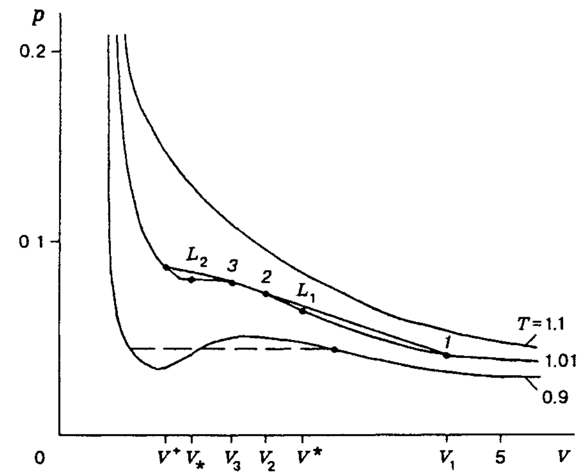

Fig. 1

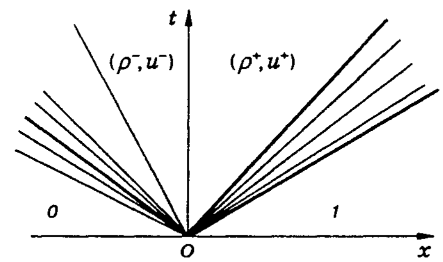

Fig. 2

coordinate $\tau_f \equiv$ const and scaling the variables q and $q \to \lambda^{-1}q$, system (2.1) is reduced to the equations of isothermal gas motion

$$V_t - u_q = 0, \quad u_t - p_q = 0 \tag{2.2}$$

with the equation of state $p = \lambda f(V, T)$. Hence, the parameter λ is insignificant in this model, and further analysis can be performed for $\lambda = 1$, i.e., for gas dynamics equations (2.1). In this case, the incompressible component is used only to substantiate that the motion under consideration is isothermal.

If the temperature T is close to the critical temperature T_c, anomalous thermodynamics properties appear in the medium. The gas behavior near the critical point can be determined by the reduced van der Waals equation [9]

$$\bar{p} = \frac{\overline{T}}{\overline{V} - 1/3} - \frac{9}{8\overline{V}^2}, \tag{2.3}$$

with quantities put into dimensionless form via the corresponding critical values: $\bar{p} = p/p_c$, $\overline{T} = T/T_c$, $\overline{V} = V/V_c$. In what follows the bar above a dimensionless variable is omitted.

For $T > 1$ dependence(2.3) is monotonic ($T = 1.1$ in Fig. 1), but for values of T close enough to unity ($1 < T < T_*$, $T_* \simeq 1.07$), the second derivative f''_{VV} becomes positive in the neighborhood of the point $V = 1$. Figure 1 shows the dependence $p = f(V, T)$ for $T = 1.01$. For $T < 1$ the dependence of the pressure on the specific volume V is already nonmonotonic ($T = 0.9$ in Fig. 1), and stable states of the medium are realized in two intervals (where $f'_V < 0$) corresponding to the liquid and gas phase.

In this article we restrict our attention to the case of $T > 1$ without phase transitions. The analysis given below is in fact applicable to a wider class of isothermal (barotropic) media with a nonconvex monotone (but not necessarily strictly monotone) equation of state. Therefore, the Maxwell rule also allows us to describe the wave structure in the neighborhood of a channel contraction for $T < 1$ (Fig. 1, $T = 0.9$).

3. The Self-Similar Solutions of System (2.1). Let us consider the problem of a sudden local contraction of a channel in which an incompressible liquid flows at a constant velocity. It is rather difficult to investigate completely the nonsteady wave picture in a channel of variable cross-section even in the case of one-dimensional model (2.1). But for large times (or small obstacle sizes) the solutions of system (2.1) behave as self-similar ones. In this case a flow regime is attained in the channel contraction which is close to steady. Thus, the structure of nonsteady longwave disturbances generated by a local obstacle can be completely inferred from an analysis of self-similar solutions of the following generalized problem of arbitrary discontinuity decay.

Let the piecewise constant initial data be specified for $t = 0$ in a channel of constant cross-section A_0.

$$(\rho(0,x), u(0,x)) = \begin{cases} (\rho_0, u_0), & x < 0, \\ (\rho_1, u_1), & x > 0. \end{cases} \tag{3.1}$$

A local channel contraction of minimum cross section $x = 0$ is located at the point $A_m < A_0$. We search for a solution in the class of self-similar solutions that depend only on the variable $\xi = x/t$ and contain a finite number of discontinuity lines. The Hugoniot conditions

$$\mathcal{D}[\rho] = [\rho u], \quad \mathcal{D}[\rho u] = [\rho u^2 + p] \tag{3.2}$$

and the Oleinik stability conditions [11] are fulfilled on the discontinuity lines $x = \mathcal{D}t$. Here $[h] = h(t, x + 0) - h(t, x - 0)$ is the jump on a discontinuity line. If for $t > 0$ the states to the left and to the right of an obstacle coincide, then the Cauchy problem (2.1), (3.1) is a classical problem on arbitrary discontinuity decay. Its solution is given in [4–7] for nonconvex equations of state. The structure of the wave adiabat, i.e., the set of admissible states (ρ, u) that can be connected to a given state (ρ_1, u_1) by a number of simple centered and stable shock waves of the same family, can be completely specified on the plane (V, p).

Let us consider the process of constructing the wave adiabat for a nonconvex equation of state (2.3) ($1 < T < T_*$) in more detail. Let V_* and V^* be points of inflection of the function $p = f(V)$, $V_* < V^*$ (Fig. 1), and $V_1 > V^*$. For $V^* < V < V_1$ the function $f(V)$ is convex and the states $(V, f(V))$ and (V_1, p_1) are connected by a stable shock transition. If $V < V^*$ but the entire graph of the function $p = f(z)$ for $V < z < V_1$ lies under a straight line L_1 passing through points $(V, f(V))$ and (V_1, p_1), then the dependence $\mathcal{D} = \mathcal{D}(V)$ is monotonic in this interval, and the Oleinik stability condition is fulfilled. Thus, for waves propagating to the right, the discontinuity velocity $\mathcal{D} = u_1 + V_1\sqrt{(f(V) - p_1)/(V_1 - V)}$ is a decreasing function of V. At the point V_2, where the straight line L_1 is tangential (Fig. 1), the velocity $\mathcal{D}(V)$ attains the local maximum $\mathcal{D}_E$, and as V decreases further, the centered compression wave, for which the Riemann invariant $s = u - \sigma(\rho)$, where $\sigma'(\rho) = c/\rho$, $c^2 = p'(\rho)$, is constant, adjoins the shock wave of maximum amplitude.

For $V < V_*$ the centered compression wave is followed by a shock transition from $(V_3, f(V_3))$ to the final state $(V, f(V))$. The point V_3 is found on the condition that the straight line L_2 is tangential to the graph of the function $p = f(V)$ (Fig. 1). The corresponding solution on the plane (t, x) is shown in Fig. 2. Let us note that the points V_2 and V_3 lie in the interval (V_*, V^*). As V decreases further, the velocity $\mathcal{D}_2$ of the rear shock wave increases, and the configuration, which involves two shock waves separated by a centered compression wave for $\mathcal{D}_2 = \mathcal{D}_E$, is transformed to a single-wave configuration. For larger amplitudes, the shock transition which connects the points $(V, f(V))$ and $(V_1, f(V_1))$ becomes stable again.

We shall not construct the wave adiabat for $V > V_1$. Note only that for $V_1 < V_*$ the wave configuration generally consists of two centered rarefaction waves separated by a shock rarefaction wave. "Splitting" of shock waves and generation of rarefaction shock waves in gases in the neighborhood of the critical point was first experimentally discovered in [12].

The solution of the problem (2.1) and (3.1) of arbitrary discontinuity decay is found from the intersection of the (p, u) diagrams for waves propagating to the right and to the left and passing through states 1 and 0, respectively [11]. But the statement of the problem essentially changes when the states of flow on each side of the local channel contraction are different.

4. Regular Regimes of Flow Through a Channel Contraction. The states $(\rho^\pm, u^\pm)$ ahead of and behind an obstacle are not arbitrary. The relationships on the discontinuity $x = 0$ are found from an analysis of the possible steady flows in the channel. Let the liquid particles run into an obstacle from the interior of the domain $x > 0$, i.e., $u^+ < 0$. Then, if the obstacle generates a wave upstream ($x > 0$), the flow ahead of the obstacle is subsonic ($u^+ + c^+ > 0$). This follows from the stability condition for shock waves and the well-known fact that the c^+ characteristics are the boundaries for simple waves propagating to the left. The state (ρ^-, u^-) can be found from the following relations for continuous stationary solutions of system (2.1)

$$\rho A u = \rho^+ A_0 u^+ = Q^+, \quad \frac{1}{2}u^2 + i(\rho) = \frac{1}{2}(u^+)^2 + i(\rho^+) = J^+ \tag{4.1}$$

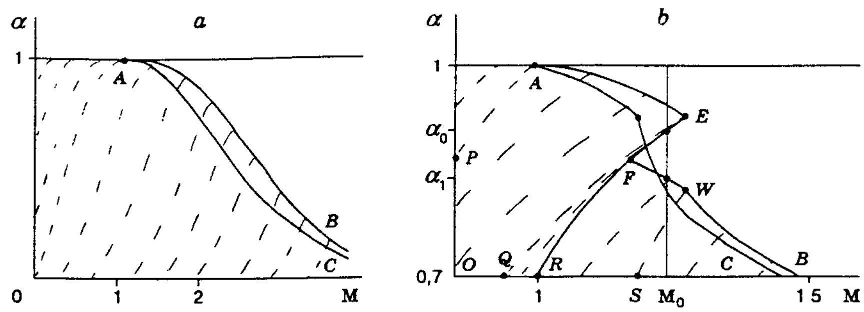

Fig. 3

in the channel contraction. Here A is the square of the channel's cross section; the function $i(\rho)$ is determined by the condition $i'(\rho) = c^2/\rho$. It is well-known for convex equations of state $[f''(V) > 0]$ that for (ρ^+, u^+) the only solution of system (4.1) different from $A = A_0$ is the supersonic state (ρ^-, u^-). In this case the flow through the channel's minimum cross-section $A = A_m$ must be sonic:

$$u_m + c_m = 0. \tag{4.2}$$

Equations (4.1) and (4.2) give us an additional relationship between the flow parameters and the relative closure of the channel $\alpha = A_m/A_0$. This relationship allows us to derive the solution of problem (2.1) and (3.1) in the domain $x > 0$ without invoking the initial data (ρ_0, u_0) and after that the solution of the Cauchy problem with data on the rays $(t = 0,\ x < 0)$ and $(t > 0,\ x = 0)$. The solution of the latter problem is identical to the classical Riemann problem considered in Section 3.

Hence, for the class of flows called regular, the steady subsonic flow transforms to supersonic ahead of the channel contraction and the downstream disturbance no longer has influence on the domain $x > 0$. Of course, when solving the Cauchy problem for system (2.1) in the domain $x < 0$, shock waves propagating at velocity $\mathcal{D} > 0$ may arise due to the nonlinearity of the equations, and the complete solution of problem (2.1) and (3.1) will be inconsistent with the regular solution in the domain $x > 0$. In this case an obstacle no longer has control over the flow upstream, and the problem is reduced to the classical problem on arbitrary discontinuity decay (2.1) and (3.1).

If the density ρ_1 is fixed, then the regular flow regime is determined by two parameters: α and $\mathrm{M} = -u_1/c_1$ $(u_1 < 0)$. The region of regular regimes in the plane (M, α) can be found in the following way. For an arbitrary wave propagating to the right and transforming the state (ρ_1, u_1) to (ρ^+, u^+), the admissible values (M, α) are determined by the condition $u^+ < 0 < u^+ + c^+$ and relationships (4.1) and (4.2), whence the dependence $\alpha = \alpha(\mathrm{M})$ can be found.

These dependences are shown in Fig. 3 by dashed lines. Figure 3a corresponds to the van der Waals gas (2.3) with a convex isotherm $(T = 1.1)$. The flows with a centered rarefaction wave moving upstream are found to the left of the curve OA. The shock wave propagates in the region OAB ahead of the obstacle. The boundary AB of the region of regular flow regimes is found to be the limiting case of zero-velocity flow of an outgoing shock wave. Only supersonic regimes of flow past an obstacle that are free from upstream disturbances can be found above the curve AB. On the other hand, there exists completely supersonic flow through the channel contraction in the region above the curve AC. The flow becomes sonic in the neighborhood of the minimum cross-section AC on A_m, and only flow with an outgoing shock wave can be found below this curve. Since the curve AC lies below the curve AB, as shown in Fig. 3a, the flow configuration in the region BAC is not uniquely specified.

A similar problem of flow nonuniqueness also arises in shallow water equations [2]. Formally the equations of one-layer shallow water are included in the class of systems (2.1), but the control conditions upstream for flows above an uneven bottom differ from (4.1) and (4.2). Nevertheless, there is a profound analogy between this class of flows and the gas dynamics equations for a channel of variable cross-section. The wave structure in one-layer shallow water ahead of an obstacle corresponds to gas flow with a convex equation of state, and anomalous regimes of gas flow with a nonconvex equation of state through a channel contraction are similar, as shown below, to the corresponding regimes of two-layer shallow water flow [3].

5. Anomalous Regimes of Gas Flow with a Nonconvex Equation of State. Let $1 < T < T_*$. In this case equation of state (2.3) has two points of inflection, and shock wave "splitting" takes place in a definite interval of values of V, as noted above. The existence of a local maximum of shock-wave propagation velocity with respect to amplitude substantially changes the region of regular flow regimes in the (M, α)-plane. Just as for convex equations of state, the dependence $\alpha = \alpha(\mathrm{M})$ can be found from relations (4.1) and (4.2) for an arbitrary state (ρ^+, u^+) connected to (ρ_1, u_1) by a combination of a simple wave and a shock wave propagating to the right. In this case $u^+ < 0 < u^+ + c^+$. Let $V_1 > V^*$, as shown in Fig. 1. A fragment of the corresponding diagram of flow regimes for $(T = 1.01)$ is presented in Fig. 3b. The region of regular flow regimes is bounded on the right by the curve *AEFWB*. A centered rarefaction wave $(V^+ > V_1)$ arises ahead of the obstacle to the left of the curve *AP*. The flow regime with an outgoing shock wave is represented in the region *OPAEQ*, with the shock wave velocity $(V_2 < V^+ < V_1)$ vanishing on the line *AE*, as in the case of a convex equation of state $T \geqslant T_*$. The point *E* corresponds to state 2 in the (V, p) diagram (see Fig. 1).

As the amplitude increases further, the disturbance's leading edge moves at the maximum velocity $\mathcal{D}_E$ and is followed by a simple wave whose trailing edge moves at the velocity $\lambda^+ = u^+ + c^+$ (the region *QER*). By virtue of $\lambda^+ > 0$ the admissible Mach numbers of the upstream flow decrease, and the line with M is the boundary $u^+ + c^+ = 0$ of the region. The configuration of two shock waves separated by a centered simple wave corresponds to the region *RFWS*. And finally, the shock waves merge on the line *WS*, and we have a single shock wave in the region *SWB* ahead of the obstacle.

It should be noted that in searching for the dependence $\alpha = \alpha(\mathrm{M})$, on the strength of (4.1) and (4.2) we assume the existence of a continuous stationary solution of (4.1) connecting the subsonic and supersonic flow on each side of a local channel contraction. But, generally speaking, this is not true in the case of nonconvex equations of state for contractions such as a Laval nozzle with a unique minimum cross-section. Steady flow regimes of dense gases with a nonconvex equation of state through a Laval nozzle have been studied in [8–10]. It has been shown that a shockless transition from subsonic to supersonic flow through the range of values of V, where $f''(V) < 0$, is possible only for a nozzle possessing several local cross-section extrema. For a contraction with a single minimum, in a stationary solution shock waves can arise in both the expanding and the contracting parts of the channel, and with passage through them relations (4.1) are only approximately fulfilled. This approximation, also used in [8–10], allows the stationary flow inside the contraction to be found on the condition that the flow through the minimum section of the channel is sonic. But the analysis of stationary solutions with arbitrary data upstream does not enable us to find the anomalous flow regimes discussed below.

The presence of the lacuna *EFW* on the boundary of the region of regular flow regimes reflects the fact that a change in a relative contraction α in some interval does not affect the wave configuration upstream, and in this case the minimum cross-section is no longer controlling. In fact, let $\mathrm{M}_F < \mathrm{M}_0 < \mathrm{M}_E$ (Fig. 3b). For decreasing α and M_0 fixed, the point (M_0, α) falls on the boundary *FE* of the domain of regular flows for $\alpha = \alpha_0$. Since $u^+ + c^+ = 0$ therewith, the flow ahead of an obstacle is critical. The dependence $\alpha = \alpha(\rho)$ derived from (4.1) is shown in Fig. 4.

As α decreases further, the wave configuration ahead of an obstacle does not change, since the boundary of the centered compression wave reaches the obstacle. Thus, ρ^+ is no longer dependent on α, and possible configurations of the flow through the contraction can be found according to Fig. 4 (for $\alpha < \alpha_0$). The regular continuous steady flow connects the state $\alpha = \alpha_0$ to the state (ρ^+, u^+) for (ρ^-, u^-) and corresponds to curve 2 (subsonic flow) in the contracting part of the channel and to curve 1 (supersonic flow) in its expanding part. When $\alpha < \alpha_0$, such a flow is impossible, and the supersonic flow corresponding to curve 3 originates in the contracting part. Theoretically, the flow down this branch can return to the initial state $(\rho^-, u^-) = (\rho^+, u^+)$ downstream and become symmetric and totally supercritical. But another type of flow is found to be more stable. In the expanding part of the channel, this type contains a shock transition from the supersonic flow on branch 3 to the sonic flow to the minimum point on curve 1. In the considered approximation ($J^+ \equiv$ const on discontinuity lines), the resulting configuration is shown by arrows in Fig. 4. This solution is represented

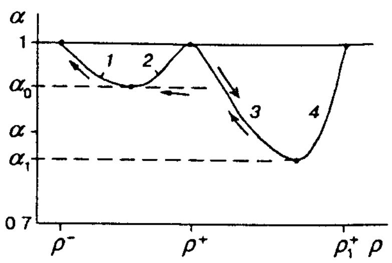

Fig. 4

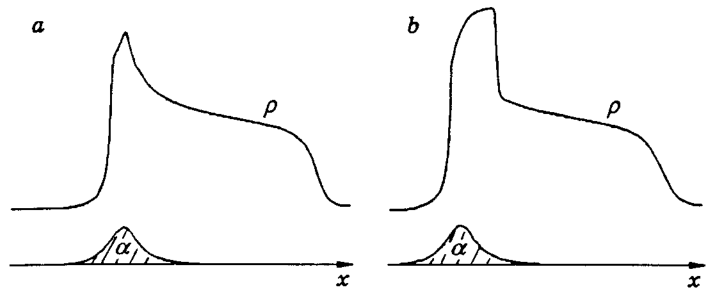

Fig. 5

for $\alpha_0 > \alpha > \alpha_1$. When $\alpha = \alpha_1$, the point (M_0, α) falls on the boundary FW of the regular flow region again.

The value of α_1 corresponds to the minimum of curves 3 and 4 in Fig. 4. When $\alpha = \alpha_1$, a shock wave transforming (ρ^+, u^+) to (ρ_1^+, u_1^+) forms ahead of an obstacle, and the regular subsonic regime of the flow along curve 4 is realized in the contracting part of the channel. A supersonic flow along branch 3 can arise in the expanded part of the channel, but, as in the previous case, the shock transition to sonic and then to supersonic flow along curve 1 for $\alpha = \alpha_0$ (shown by the arrows directed to the left) is found to be more stable.

With intersection of the line FW, the peculiarities of the change in the flow regime as the minimum cross-section of the channel decreases can be illustrated by numerical solution of the nonstationary problem of the sudden contraction of a channel.

Figure 5 presents the distribution of the density ρ in the neighborhood of the channel contraction for a van der Waals gas with equation of state (2.1) for $T = 1.01$, $\rho_0 = 0.3$. For $\alpha = A_m/A_0$ [$A_m = \min A(x)$] the channel's profile is given by the function $y = 1 - A(x)/A_0$, which bounds the shaded region. The Mach number of the upstream flow M_0 equals 1.15 in either case. When $\alpha = 0.8$ (Fig. 5b), the point (M_0, α) is in the region EFW, and supersonic flow through the channel contraction is realized. The flow is symmetric in the neighborhood of the minimum contraction, and a shock wave transforming the supersonic flow to sonic arises downstream. We did not use the shock wave approximation of (4.1) in the calculations. So the configuration shown in Fig. 4 qualitatively reproduces the wave picture which appears in the basic model as well. When $\alpha = 0.79$ (Fig. 5b) the flow through the contracting part of the channel is regular, and the transition from subsonic to supersonic flow takes place in the minimum cross-section of the channel. In both cases a shock wave followed by a compression wave propagates ahead of an obstacle at the velocity $\mathcal{D}_E$.

Thus, with intersection of the boundaries of the region of regular flow, a sharp change in the wave configuration takes place in the neighborhood of the channel contraction, which may be responsible for hysteresis, i.e., dependence of the solution on the prehistory of the process. Naturally, for both nonconvex and convex equations of state, a region of nonuniqueness of the flow regimes appears in the plane (M, α) for $\mathrm{M} > 1$, in which flow with an outgoing shock wave as well as supercritical flow with undisturbed flow upstream can be realized. The latter flow regime may take place above the curve AC (Fig. 3b).

6. Conclusions. (1) The structure of nonsteady waves propagating upstream from a local channel contraction has been studied. If the minimum cross-section of the channel has total control over the upstream flow, the flow behind an obstacle is supersonic, and the wave configuration downstream can be found as a solution of the Cauchy problem. For gases with a nonconvex equation of state, steady flows through the channel of variable section can contain several shock waves, as shown in [8–10] and in Figs. 4 and 5. The advantage of the nonsteady approach to the problem of wave structure in the neighborhood of a local channel contraction over the investigations of steady flows performed in [8, 9] is not only the fact that anomalous flow regimes are found in the region EFW (Fig. 3b), but also that the analysis of possible wave configurations in the contraction is simplified when we succeed in separating the shock waves generated by the obstacle itself and those generated by the downstream conditions.

(2) A particular class of isothermal motions of a van der Waals gas in the neighborhood of a critical point has been investigated, but the approach used is also applicable to arbitrary barotropic motions of the gas.

In addition, since contact discontinuities cannot propagate upstream of an obstacle in the case of nonsteady and nonisothermal flows of a Van der Waals gas, the wave structure in the neighborhood of the critical point for arbitrary gases is similar to that in the isothermal case discussed above.

(3) The existing analogy between the well-known flow regimes for a gas–liquid medium and two-layer shallow water [3] gives us the ability to state that the existence of anomalous flow regimes in the neighborhood of a channel contraction is a common property of models of two-phase flows without a monotone dependence of the propagation velocity of nonlinear disturbances on the wave amplitude.

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